Production Theory

A Complete Mathematical Guide

Cobb-Douglas · Leontief · Linear · CES

Marginal Products · MRTS · Returns to Scale · Lagrangian Cost Minimization · Conditional Factor Demands · Cost Functions · Profit Maximization · Comparative Statics · Envelope Theorem · IFT · Shephard's Lemma · Full Problem Sets with Solutions

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Part 0 · Foundations: What Is a Production Function?

§0.1 The Production Function and Isoquants

A **production function** $F(K, L)$ maps an input bundle of capital K and labour L into the *maximum* feasible output quantity q:

$$q = F(K, L) \tag{0.1}$$

The function encodes the firm's *technology* — all physically feasible ways to transform inputs into output. The **isoquant** for output level q□ is the set of all bundles producing exactly that quantity:

$$IQ(q_0) = \{ (K, L) : F(K, L) = q_0 \} \tag{0.2}$$

Isoquants are the production analogue of indifference curves. Their **shape** encodes substitutability: smooth convex curves (Cobb-Douglas, CES) allow a continuum of cost-minimising input mixes; right-angle kinks (Leontief) allow only one; straight lines (linear) make inputs perfectly interchangeable.

§0.2 Key Properties and Taxonomy

We impose four standard regularity conditions on F:

Monotonicity (free disposal): $\partial F/\partial K \geq 0$ and $\partial F/\partial L \geq 0$

Diminishing marginal returns: $\partial^2F/\partial K^2 \leq 0$ and $\partial^2F/\partial L^2 \leq 0$

Quasi-concavity: isoquants are convex toward the origin

Essentiality: $F(0, L) = F(K, 0) = 0$

Function	Form	Isoquant	Elasticity σ	RTS controlled by
Cobb-Douglas	$AK^{\alpha}L^{\beta}$	Smooth, convex	$\sigma = 1$	$\alpha + \beta$ vs 1
Leontief	$\min(K/a, L/b)$	L-shaped kink	$\sigma = 0$	Always CRS
Linear	$aK + bL$	Straight line	$\sigma = \infty$	Always CRS
CES	$A(\alpha K^{\rho}+\beta L^{\rho})^{1/\rho}$	Smooth, varies	$\sigma = 1/(1-\rho)$	Always CRS (std)

Table 0.1 — Taxonomy of production functions.

Part I · Cobb-Douglas Production Functions

§1.1 Definition, Notation & Output Elasticities

The **Cobb-Douglas** production function is the workhorse of microeconomic theory. Its log-linear multiplicative structure yields closed-form solutions to nearly every optimization problem while preserving economically sensible properties.

Key Result — Cobb-Douglas Definition

$F(K, L) = A \cdot K^{\alpha} \cdot L^{\beta}$

$A > 0$ = total factor productivity (TFP)

$\alpha > 0$ = capital output elasticity

$\beta > 0$ = labour output elasticity

The parameters α and β are output elasticities — verifiable by log-differentiation:

$$\partial \ln F / \partial \ln K = \alpha \text{ (1\% rise in K raises output by } \alpha\%)$$

$$\partial \ln F / \partial \ln L = \beta \text{ (1\% rise in L raises output by } \beta\%)$$

Intuition

Because the Cobb-Douglas is log-linear, output elasticities are constant everywhere on the isoquant map. This is both its great strength (tractability) and its empirical rigidity — real technologies may have varying elasticities.

§1.2 Marginal Products, MRTS & Diminishing Returns

Step 1 — Compute the marginal products.

Differentiate F partially with respect to each input:

$$MP_K = \partial F/\partial K = \alpha A K^{\alpha-1} L^{\beta} = \alpha \cdot (q/K) \tag{1.1}$$

$$MP_L = \partial F/\partial L = \beta A K^{\alpha} L^{\beta-1} = \beta \cdot (q/L) \tag{1.2}$$

The elegant result: $MP_K = \alpha(q/K)$, so capital's marginal product is the average product of capital scaled by α. Same for labour.

Step 2 — Verify diminishing marginal returns (SOC).

$$\partial^2F/\partial K^2 = \alpha(\alpha-1)\, AK^{\alpha-2}L^{\beta} < 0 \ \square\ \alpha < 1 \quad (1.3)$$

Provided α < 1 and β < 1, each input exhibits diminishing marginal returns *in isolation*. Note this is a property of a single input varying while the other is held fixed — distinct from returns to scale.

Step 3 — Derive the MRTS.

The **Marginal Rate of Technical Substitution** $MRTS_{LK}$ is the rate at which the firm can trade labour for capital along an isoquant while holding output constant. It equals the ratio of marginal products:

$$MRTS_{LK} = MP_L / MP_K = (\beta/\alpha) \cdot (K/L) \quad (1.4)$$

□ **Intuition**

MRTS is proportional to K/L. As the firm substitutes labour for capital (moving down the isoquant), K/L falls, so MRTS falls — this is ***diminishing MRTS****, which makes isoquants bow toward the origin. It is the production analogue of diminishing MRS in consumer theory.*

□ **Common Error**

MRTS is always |dK/dL| along the isoquant. From implicit differentiation of F(K,L) = q: dK/dL = −MP_L/MP_K. The negative sign reflects the downward slope; we report the absolute value. Do not confuse MRTS with the negative slope.

§1.3 Returns to Scale — Formal Proof

Returns to Scale (RTS) ask: if we scale *all* inputs by t > 1 simultaneously, what happens to output? This is a global proportional scaling property, distinct from the marginal product of one input.

Evaluate F(tK, tL) for arbitrary t > 1:

$F(tK, tL) = A \cdot (tK)^{\alpha} \cdot (tL)^{\beta}$

$= A \cdot t^{\alpha} K^{\alpha} \cdot t^{\beta} L^{\beta}$

$= t^{\alpha+\beta} \cdot AK^{\alpha}L^{\beta}$

$$F(tK, tL) = t^{\alpha+\beta} \cdot F(K, L) \text{ [homogeneous of degree } \alpha+\beta] \quad (1.5)$$

Condition	Returns to Scale	Meaning
$\alpha + \beta > 1$	Increasing (IRS)	Doubling inputs more than doubles output — economies of scale
$\alpha + \beta = 1$	Constant (CRS)	Doubling inputs exactly doubles output — replicable technology
$\alpha + \beta < 1$	Decreasing (DRS)	Doubling inputs less than doubles output — diseconomies of scale

Table 1.1 — Returns to scale for Cobb-Douglas.

Intuition

Under CRS ($\alpha+\beta = 1$), the firm can perfectly replicate itself: two identical plants using (2K, 2L) produce exactly 2q. CRS is the competitive equilibrium benchmark — it is why long-run supply curves are flat and why CRS firms earn zero economic profit at the optimum.

§1.4 Cost Minimization via Lagrangian

The firm's **cost minimization problem (CMP)** is to produce target output q at the lowest possible total expenditure, given factor prices w (wage) and r (rental rate of capital):

The Cost Minimization Problem

$\min_{K,L}\ C = wL + rK$

s.t. $F(K, L) = AK^{\alpha}L^{\beta} = q$

$K \geq 0,\ L \geq 0$

Step 1 — Write the Lagrangian.

Introduce multiplier λ for the equality constraint:

$$\mathscr{L}(K, L, \lambda) = wL + rK - \lambda\,[\,AK^{\alpha}L^{\beta} - q\,] \qquad (1.6)$$

Step 2 — First-Order Conditions (FOCs).

Set each partial derivative to zero:

$$\partial\mathscr{L}/\partial K = r - \lambda \cdot \alpha AK^{\alpha-1}L^{\beta} = 0 \;\square\; r = \lambda \cdot MP_K \qquad (1.7)$$

$$\partial\mathscr{L}/\partial L = w - \lambda \cdot \beta AK^{\alpha}L^{\beta-1} = 0 \;\square\; w = \lambda \cdot MP_L \qquad (1.8)$$

$$\partial\mathscr{L}/\partial \lambda = -[\,AK^{\alpha}L^{\beta} - q\,] = 0 \;\square\; F(K,L) = q \qquad (1.9)$$

Step 3 — Derive the tangency condition.

Divide (1.7) by (1.8) to eliminate λ:

$r / w = MP_K / MP_L$

$= [\,\alpha AK^{\alpha-1}L^{\beta}\,] / [\,\beta AK^{\alpha}L^{\beta-1}\,]$

$= (\alpha/\beta) \cdot (L/K)$

Rearranging:

$$K / L = (\alpha w) / (\beta r) \;\square\; K = (\alpha w/\beta r) \cdot L \qquad (1.10)$$

☐ **Intuition**

Equation (1.10) is the production analogue of the consumer's tangency condition MRS = p_x/p_y. At the optimum, the input ratio K/L depends only on relative factor prices w/r and the technology ratio α/β. It is ***independent of the output target q*** *— a hallmark of Cobb-Douglas.*

Step 4 — Solve for conditional factor demands.

Let $\Omega = \alpha + \beta$. Substitute $K = (\alpha w/\beta r)L$ into the output constraint $AK^{\alpha}L^{\beta} = q$:

$A \cdot (\alpha w/\beta r)^{\alpha} \cdot L^{\alpha} \cdot L^{\beta} = q$

$A \cdot (\alpha w/\beta r)^{\alpha} \cdot L^{\Omega} = q$

$L^{\Omega} = q / [\, A(\alpha w/\beta r)^{\alpha} \,]$

$$L^*(w, r, q) = (q/A)^{1/\Omega} \cdot (\beta r/\alpha w)^{\alpha/\Omega} \tag{1.11}$$

$$K^*(w, r, q) = (\alpha w/\beta r) \cdot L^*(w, r, q) = (q/A)^{1/\Omega} \cdot (\alpha w/\beta r)^{\beta/\Omega} \tag{1.12}$$

Common Error

Always track the $q^{1/\Omega}$ term — it carries the returns-to-scale information. Under CRS ($\Omega = 1$): demands scale linearly in q. Under IRS ($\Omega > 1$): demands are concave in q (scale economies). Under DRS: convex.

§1.5 The Cost Function

Substitute L^* and K^* back into $C = wL + rK$:

$C^*(w,r,q) = w \cdot L^* + r \cdot K^*$

$= w \cdot L^* + r \cdot (\alpha w/\beta r) \cdot L^*$

$= w \cdot L^* \cdot [\ 1 + \alpha/\beta\ ]$

$= w \cdot L^* \cdot [\ \Omega/\beta\ ]$

After collecting powers of w and r (and absorbing constants into B):

> □ **Key Result — Cobb-Douglas Cost Function**
>
> $c(w, r, q) = B \cdot w^{\beta/\Omega} \cdot r^{\alpha/\Omega} \cdot q^{1/\Omega}$
>
> where $B = (1/A)^{1/\Omega} \cdot (\beta/\Omega)^{-\beta/\Omega} \cdot (\alpha/\Omega)^{-\alpha/\Omega}$ is a technology constant.

§1.6 Shephard's Lemma & Cost Function Properties

Shephard's Lemma is the Envelope Theorem applied to the cost function: since c(w, r, q) is the minimised cost, differentiating with respect to a factor price recovers the conditional demand directly:

$$\partial c/\partial w = L^*(w, r, q) \text{ (Shephard's Lemma for labour)} \tag{1.13}$$

$$\partial c/\partial r = K^*(w, r, q) \text{ (Shephard's Lemma for capital)} \tag{1.14}$$

Verify (1.13) for Cobb-Douglas: Differentiating $c = B\, w^{\beta/\Omega}\, r^{\alpha/\Omega}\, q^{1/\Omega}$ with respect to w:

$\partial c/\partial w = B \cdot (\beta/\Omega) \cdot w^{\beta/\Omega - 1} \cdot r^{\alpha/\Omega} \cdot q^{1/\Omega}$

$= (\beta/\Omega) \cdot (c/w) = L^*$ ✓

Four key properties of c(w, r, q):

1. Homogeneous degree 1 in (w, r): $c(\lambda w, \lambda r, q) = \lambda\, c(w, r, q)$. Doubling both factor prices exactly doubles cost.

2. Increasing and concave in each factor price separately: $\partial c/\partial w > 0$, $\partial^2 c/\partial w^2 < 0$. Concavity reflects that the firm can substitute away from an increasingly expensive input.

3. Non-decreasing in q: $MC = \partial c/\partial q = (1/\Omega)\, B\, w^{\beta/\Omega}\, r^{\alpha/\Omega}\, q^{1/\Omega - 1} \geq 0$.

4. Convexity in q controls scale: Under DRS ($\Omega < 1$): $d^2c/dq^2 > 0$ (convex cost, rising MC). Under CRS ($\Omega = 1$): MC constant. Under IRS: MC falling.

§1.7 Profit Maximization & Output Supply

A competitive firm takes output price p as given and solves the two-step problem:

Two-Step Profit Maximization

Step 1 (done): solve CMP → get c(w, r, q)

Step 2: choose q to maximise

$\pi(q) = pq - c(w, r, q)$

FOC: $p = dc/dq = MC(q)$ [price equals marginal cost]

SOC: $d^2c/dq^2 > 0$ [cost must be convex in q → requires DRS]

Compute MC for Cobb-Douglas:

$$MC(q) = dc/dq = (1/\Omega) \cdot B \cdot w^{\beta/\Omega} \cdot r^{\alpha/\Omega} \cdot q^{1/\Omega - 1} \qquad (1.15)$$

Set p = MC and solve for q (DRS case, $\Omega < 1$):

$p = (1/\Omega)\, B\, w^{\beta/\Omega}\, r^{\alpha/\Omega}\, q^{1/\Omega - 1}$

$q^{1/\Omega - 1} = p\Omega / [\, B\, w^{\beta/\Omega}\, r^{\alpha/\Omega}\,]$

$$q^*(p, w, r) = [\, p\Omega / (B\, w^{\beta/\Omega}\, r^{\alpha/\Omega})\,]^{\Omega/(1-\Omega)} \text{ [output supply, DRS]} \qquad (1.16)$$

□ **Common Error**

Under CRS ($\Omega = 1$): MC is constant in q, so $\pi(q) = (p - MC)q$ is linear. The firm wants $q \to \infty$ if $p > MC$, $q = 0$ if $p < MC$, and is indeterminate if $p = MC$. No finite interior maximum exists. CRS firms require the zero-profit condition $p = \min(AC)$ to pin down equilibrium.

SOC verification:

$$d^2c/dq^2 = (1/\Omega)(1/\Omega - 1) \cdot B \cdot w^{\beta/\Omega}\, r^{\alpha/\Omega} \cdot q^{1/\Omega - 2} > 0 \;\square\; \Omega < 1 \text{ (DRS)} \qquad (1.17)$$

§1.8 Comparative Statics — Envelope Theorem & IFT

Two tools generate comparative static results without fully re-solving a model:

Tool A — Envelope Theorem. Let $V(p) = \max_q\{pq - c(w,r,q)\}$ be the profit function. The Envelope Theorem states:

$$dV/dp = \partial\pi/\partial p\,|_{q=q^*} = q^* \qquad (1.18)$$

The indirect channel through q^* vanishes because at q^*, $\partial\pi/\partial q = p - MC(q^*) = 0$ by the FOC. So the derivative of the maximised value with respect to a parameter equals the direct partial — no re-optimisation required.

□ **Intuition**

The Envelope Theorem says: at an optimum, the first-order effect of changing a parameter on the optimal objective equals the direct partial effect. The indirect channel (how the optimizer adjusts) contributes zero, because the FOC sets that gradient to zero by construction. This eliminates enormous algebra on exams.

Tool B — Implicit Function Theorem (IFT) for Market Equilibrium. A per-unit tax t shifts firm supply. Equilibrium price p^* satisfies:

$$G(p^*, t) \equiv X^D(p^*) - Y^S(p^*, t) = 0 \qquad (1.19)$$

Differentiate totally with respect to t:

$(\partial G/\partial p^*) \cdot (dp^*/dt) + \partial G/\partial t = 0$

$$dp^*/dt = -(\partial G/\partial t) / (\partial G/\partial p^*) = (\partial Y^S/\partial t) / (dX^D/dp - dY^S/dp) \qquad (1.20)$$

□ **Key Result — Tax Incidence Formula**

Let $\varepsilon^D = (p/X^D)(dX^D/dp) < 0$ and $\varepsilon^S = (p/Y^S)(dY^S/dp) > 0$.

$dp^*/dt = \varepsilon^S / (\varepsilon^S - \varepsilon^D) \in (0, 1)$

As $\varepsilon^D \to 0$ (perfectly inelastic demand): $dp^*/dt \to 1$ (100% on consumers)

As $\varepsilon^S \to 0$ (perfectly inelastic supply): $dp^*/dt \to 0$ (100% on producers)

Part II · Leontief (Perfect Complements)

§2.1 Definition and Isoquant Geometry

The **Leontief** function models technologies with zero substitutability — inputs must be used in fixed proportions.

□ Key Result — Leontief Production Function

$F(K, L) = \min(K/a , L/b)$

$a > 0$ = capital required per unit of output

$b > 0$ = labour required per unit of output

□ Intuition

A taxi firm needs exactly 1 car (K) and 1 driver (L) per trip. An extra car without a driver produces zero additional trips; an extra driver without a car similarly contributes nothing. Inputs are perfect complements that must be used in the ratio a:b.

The isoquant for output q_0 is L-shaped with its kink at the efficient bundle:

$$\text{Kink point: } K = a \cdot q_0 , L = b \cdot q_0 \quad (2.1)$$

On the vertical arm ($K > aq_0$, $L = bq_0$), capital is wastefully abundant. On the horizontal arm, labour is the binding constraint. The MRTS is 0 on the horizontal arm, ∞ on the vertical arm, and undefined at the kink.

§2.2 Cost Minimization — No Lagrangian Interior Solution

F is not differentiable at the kink, so the Lagrangian FOC approach fails. Instead, reason directly: any point *off* the kink along the isoquant has one input in excess. Reducing that input saves cost without reducing output. Therefore the optimum is always the kink.

$$K^*(w, r, q) = a \cdot q \quad (2.2)$$

$$L^*(w, r, q) = b \cdot q \quad (2.3)$$

Key observation: conditional demands are **linear in q** and **independent of prices** (w, r) — there is no substitution effect whatsoever.

□ Key Result — Leontief Cost Function

$c(w, r, q) = wL^* + rK^* = (wb + ra) \cdot q$

→ Linear in q ☐ constant MC = wb + ra = AC

→ CRS always ☐ $\Omega = 1$ regardless of a and b

→ No factor substitution ☐ $\partial L^*/\partial r = 0$ and $\partial K^*/\partial w = 0$

Shephard's Lemma check:

$\partial c/\partial w = b \cdot q = L^*$ ✓

$\partial c/\partial r = a \cdot q = K^*$ ✓

□ Common Error

For Leontief, the cross-price effects $\partial L^*/\partial r = 0$ and $\partial K^*/\partial w = 0$ — inputs are neither substitutes nor complements in the cost-minimisation sense. Contrast with Cobb-Douglas where $\partial L^*/\partial r > 0$ (inputs are substitutes): a rise in r makes labour relatively cheaper, so the firm substitutes toward labour.

Part III · Linear (Perfect Substitutes)

§3.1 Definition and Corner Solutions

The **linear** production function represents technologies where capital and labour are perfect substitutes at a constant rate.

□ Key Result — Linear Production Function

$F(K, L) = aK + bL$

$MP_K = a$ (constant), $MP_L = b$ (constant)

$MRTS_{LK} = b/a$ (constant everywhere)

Isoquants are straight downward-sloping lines

Because the MRTS b/a is constant, the optimal solution is determined by comparing MRTS to the input price ratio w/r:

Condition	Optimal Bundle	Economic Logic
$w/r < b/a$ (labour cheap)	$L^* = q/b$, $K^* = 0$	Output per dollar higher for labour
$w/r > b/a$ (capital cheap)	$K^* = q/a$, $L^* = 0$	Output per dollar higher for capital
$w/r = b/a$ (indifferent)	Any (K,L) with $aK+bL = q$	Infinitely many optima along isoquant

Table 3.1 — Corner solutions for linear production.

□ Key Result — Linear Cost Function

$c(w, r, q) = \min(w/b , r/a) \cdot q$

Case $w/r < b/a$: $c = (w/b) \cdot q$ [only labour used]

Case $w/r > b/a$: $c = (r/a) \cdot q$ [only capital used]

□ Intuition

The linear cost function embodies winner-take-all logic: the firm uses only the cheaper input (per unit of output). Factor price changes that cross the threshold b/a cause the firm to switch entirely from one input to the other — a discontinuous comparative static. Pure perfect substitutes are the limiting case of the CES as $\rho \to 1$.

Part IV · CES Production Functions

§4.1 Definition & the Elasticity of Substitution σ

The **Constant Elasticity of Substitution (CES)** function nests Cobb-Douglas, Leontief and the linear case as special limits, parameterised by the substitution parameter ρ.

□ Key Result — CES Production Function

$F(K, L) = A [\alpha K^{\rho} + \beta L^{\rho}]^{1/\rho}$

$A > 0$ (TFP), $\alpha, \beta > 0$ with $\alpha + \beta = 1$ (distribution params)

$\rho \in (-\infty, 1) \setminus \{0\}$ is the substitution parameter

Elasticity of substitution: $\sigma = 1 / (1 - \rho)$

§4.2 Special Cases — The Unified Family

Limit / ρ	Resulting Function	σ	Interpretation
$\rho \to 1$	Linear $F = A(\alpha K + \beta L)$	$\sigma \to \infty$	Perfect substitutes
$\rho \to 0$	Cobb-Douglas $F = AK^{\alpha}L^{\beta}$	$\sigma = 1$	Unit-elastic
$\rho \to -\infty$	Leontief $F = A\cdot\min(\alpha K, \beta L)$	$\sigma \to 0$	Perfect complements
$\rho = -1$	$F = A(\alpha K^{-1}+\beta L^{-1})^{-1}$	$\sigma = 1/2$	Harmonic mean
ρ □ (0,1)	CES, elastic substitution	$\sigma > 1$	High substitutability
ρ □ (−∞,0)	CES, inelastic substitution	$\sigma < 1$	Low substitutability

Table 4.1 — CES as a unified family.

□ Intuition

The CES is the Swiss Army knife of production functions. By varying ρ, the economist continuously tunes substitutability. Cobb-Douglas (σ = 1) is the interior point of the full spectrum — a natural benchmark because input shares are constant and exactly equal α and β.

§4.3 Marginal Products, MRTS & Factor Demands

Let $\Phi = \alpha K^{\rho} + \beta L^{\rho}$ for compactness. Then $F = A\Phi^{1/\rho}$.

Marginal product of K:

$MP_K = \partial F/\partial K = A \cdot (1/\rho) \cdot \Phi^{1/\rho - 1} \cdot \rho \cdot \alpha K^{\rho-1}$

$= \alpha A \cdot K^{\rho-1} \cdot \Phi^{1/\rho - 1}$

$= \alpha \cdot (F/K)^{1-\rho}$ [simplified]

$$MP_K = \alpha \cdot (F/K)^{1-\rho} \quad (4.1)$$

MRTS for CES:

$$MRTS_{LK} = MP_L/MP_K = (\beta/\alpha) \cdot (K/L)^{1-\rho} \quad (4.2)$$

Check limits: as $\rho \to 0$, $(K/L)^{1-\rho} \to K/L$ — recovering the Cobb-Douglas MRTS $(\beta/\alpha)(K/L)$. As $\rho \to 1$, MRTS $\to \beta/\alpha$ = constant, matching the linear case.

Formal derivation of σ:

Take logs of $MRTS = (\beta/\alpha)(K/L)^{1-\rho}$:

$\ln MRTS = \ln(\beta/\alpha) + (1-\rho) \ln(K/L)$

$\partial \ln(K/L) / \partial \ln MRTS = 1/(1-\rho) = \sigma$ ✓

$$\sigma = 1/(1 - \rho) \text{ [constant along the entire isoquant]} \quad (4.3)$$

Cost minimization — tangency condition:

Set MRTS = w/r:

$(\beta/\alpha)(K/L)^{1-\rho} = w/r$

$(K/L)^{1-\rho} = (\alpha/\beta)(w/r)$

$K/L = [(\alpha/\beta)(w/r)]^{1/(1-\rho)} = (\alpha/\beta)^{\sigma} \cdot (w/r)^{\sigma}$

$$K^*/L^* = (\alpha/\beta)^{\sigma} \cdot (w/r)^{\sigma} \quad (4.4)$$

□ Key Result — CES Cost Function

$c(w, r, q) = (1/A) [\alpha^{\sigma} w^{1-\sigma} + \beta^{\sigma} r^{1-\sigma}]^{1/(1-\sigma)} \cdot q$

Verify limiting cases:

$\sigma \to 1$ (Cobb-Douglas): $c \to (1/A) w^{\beta} r^{\alpha} q$ [up to constant]

$\sigma \to 0$ (Leontief): $c \to (w\beta + r\alpha)/A \cdot q$ [linear in prices]

$\sigma \to \infty$ (Linear): $c \to \min(w/b, r/a) \cdot q$

Part V · Problem Set

Show all derivations in full. State every FOC explicitly. Verify SOCs. Apply the Envelope Theorem or IFT wherever indicated.

Problem 1

(a) A firm has $F(K, L) = K^{1/3}L^{2/3}$. Prices are w and r.

(b) Set up the Lagrangian CMP. Derive all three FOCs.

(c) Solve for conditional factor demands $K^*(w, r, q)$ and $L^*(w, r, q)$.

(d) Derive the cost function $c(w, r, q)$. Compute MC and AC. What are the returns to scale?

(e) Show MC = AC for all q. Explain why this firm has no well-defined competitive supply function.

Problem 2

(a) A firm has $F(K, L) = (K^{-1} + L^{-1})^{-1}$ [CES with $\rho = -1$, $\sigma = 1/2$].

(b) Compute MP_K and MP_L. Derive MRTS and verify it is diminishing.

(c) Solve the CMP. Find conditional demands $K^*(w, r, q)$ and $L^*(w, r, q)$.

(d) Derive $c(w, r, q)$. Verify Shephard's Lemma for both inputs.

(e) Show that as $\sigma \to 0$ your cost function converges to the Leontief form.

Problem 3

(a) A competitive firm has cost function $c(w, r, q) = w^{2/3}r^{1/3}q^2$.

(b) Derive output supply $q^*(p, w, r)$ from the FOC. Check the SOC.

(c) Derive the profit function $\pi^*(p, w, r) = \max_q\{pq - c\}$.

(d) Verify Hotelling's Lemma: $\partial\pi^*/\partial p = q^*$.

(e) Derive unconditional factor demands using the Envelope Theorem ($\partial\pi^*/\partial w = -L$, $\partial\pi^*/\partial r = -K$).

(f) A per-unit tax t replaces p with $(p-t)$. Use the IFT to find dq^*/dt and sign it.

Problem 4

(a) A firm has $F(K, L) = AK^{\alpha}L^{\beta}$ with $\alpha + \beta < 1$ (DRS).

(b) Derive conditional and unconditional factor demands. Show unconditional demands are homogeneous degree 0 in (p, w, r).

(c) Prove $\pi^*(p, w, r)$ is homogeneous degree 1 in (p, w, r) using the Euler homogeneous function theorem.

(d) Compute $\partial\pi^*/\partial p$ and $\partial\pi^*/\partial w$ and verify both Envelope Theorem applications.

(e) Edge case: what happens to K^*/L^* as $w/r \to \infty$? As $w/r \to 0$? Interpret economically.

Problem 5

(a) A firm has $F(K, L) = \min(2K, 3L)$ [Leontief].

(b) Find the cost-efficient K/L ratio. Write down $c(w, r, q)$.

(c) Verify Shephard's Lemma: $\partial c/\partial w = L^*$ and $\partial c/\partial r = K^*$.

(d) A capital subsidy reduces effective rental to $(r - s)$. Find $K^*(w, r-s, q)$ and use the IFT to derive dK^*/ds. What is surprising?

(e) Contrast part (d) with the Cobb-Douglas response to the same subsidy. Explain the fundamental difference.

Problem 6

(a) [Exam level] A firm has $F(K, L; p) = \ln K + L + \gamma K \ln p$ where p is output price and $\gamma > 0$. Price enters the production function directly.

(b) Compute MRTS and show it depends on p.

(c) Set up the CMP with constraint $F(K, L; p) = q$. Derive FOCs and solve for $K^*(w, r, q; p)$ and $L^*(w, r, q; p)$.

(d) Apply the Envelope Theorem to $c(w, r, q; p)$ to find $\partial c/\partial p$. Interpret this derivative.

(e) Edge case: as $p \to \infty$, what happens to K^*/L^*? As $p \to 0+$? Verify with your closed-form demands.

Part VI · Full Worked Solutions

Solution — Problem 1 [$F(K,L) = K^{1/3}L^{2/3}$]

(a) & (b) Lagrangian and conditional demands.

Here $\alpha = 1/3$, $\beta = 2/3$, $A = 1$, $\Omega = 1$.

$$\mathscr{L}(K, L, \lambda) = wL + rK - \lambda [K^{1/3}L^{2/3} - q]$$

FOCs:

$\partial\mathscr{L}/\partial K = r - (1/3)\lambda K^{-2/3}L^{2/3} = 0$ ☐ $r = (1/3)\lambda(L/K)^{2/3}$

$\partial\mathscr{L}/\partial L = w - (2/3)\lambda K^{1/3}L^{-1/3} = 0$ ☐ $w = (2/3)\lambda(K/L)^{1/3}$

$\partial\mathscr{L}/\partial\lambda = -[K^{1/3}L^{2/3} - q] = 0$

Divide FOC-K by FOC-L to eliminate λ:

$r/w = (1/2)(L/K)$ ☐ $K/L = w/(2r)$ ☐ $K = (w/2r)L$

Substitute into constraint:

$(w/2r)^{1/3}L^{1/3} \cdot L^{2/3} = q$

$(w/2r)^{1/3} \cdot L = q$

$$L^*(w, r, q) = q \cdot (2r/w)^{1/3} \quad K^*(w, r, q) = q \cdot (w/2r)^{2/3}$$

(c) & (d) Cost function, MC, AC and CRS.

$C^* = wL^* + rK^* = w\cdot q(2r/w)^{1/3} + r\cdot q(w/2r)^{2/3}$

$= q \cdot w^{2/3}r^{1/3} \cdot [2^{1/3} + (1/2)^{2/3}]$

$= q \cdot w^{2/3}r^{1/3} \cdot (3/2^{2/3})$ [since $2^{1/3} + 2^{-2/3} = 3\cdot 2^{-2/3}$]

$$c(w, r, q) = (3 / 2^{2/3}) \cdot w^{2/3} r^{1/3} \cdot q$$

$MC = dc/dq = (3/2^{2/3}) w^{2/3}r^{1/3} = AC$ (linear in q ☐ constant MC = AC).

With $\Omega = 1$, this is CRS — as expected.

(e) No well-defined supply under CRS.

$\pi(q) = pq - MC\cdot q = (p - MC)\cdot q$ is linear in q. If $p > MC$ the firm wants $q \to \infty$; if $p < MC$ it wants $q = 0$; if $p = MC$ it is indifferent over all $q \geq 0$. No finite profit-maximising quantity exists. Equilibrium requires the zero-profit condition $p = \min(AC) = MC$.

Solution — Problem 3 [$c(w, r, q) = w^{2/3}r^{1/3}q^2$]

(a) Output supply and SOC.

$\pi(q) = pq - w^{2/3}r^{1/3}q^2$

FOC: $p = 2w^{2/3}r^{1/3}q$

$$q^*(p, w, r) = p / [\ 2w^{2/3}r^{1/3}\ ]$$

SOC: $d^2\pi/dq^2 = -2w^{2/3}r^{1/3} < 0$ ✓

(b) & (c) Profit function and Hotelling's Lemma.

$\pi^* = p \cdot q^* - w^{2/3}r^{1/3}(q^*)^2$

$= p \cdot [\ p/(2w^{2/3}r^{1/3})\ ] - w^{2/3}r^{1/3} \cdot p^2/(4w^{4/3}r^{2/3})$

$= p^2/(2w^{2/3}r^{1/3}) - p^2/(4w^{2/3}r^{1/3})$

$$\pi^*(p, w, r) = p^2 / [\ 4w^{2/3}r^{1/3}\ ]$$

Hotelling's Lemma: $\partial\pi^*/\partial p = 2p/(4w^{2/3}r^{1/3}) = p/(2w^{2/3}r^{1/3}) = q^*$ ✓

(d) Unconditional factor demands via Envelope Theorem.

$L(p,w,r) = -\partial\pi^*/\partial w = p^2(2/3) / [4w^{5/3}r^{1/3}] = p^2 / [6w^{5/3}r^{1/3}]$

$K(p,w,r) = -\partial\pi^*/\partial r = p^2(1/3) / [4w^{2/3}r^{4/3}] = p^2 / [12w^{2/3}r^{4/3}]$

(f) Comparative static dq*/dt via IFT.

Replace p with (p − t). FOC: $G(q, t) = (p-t) - 2w^{2/3}r^{1/3}q = 0$

$\partial G/\partial q = -2w^{2/3}r^{1/3} < 0$

$\partial G/\partial t = -1 < 0$

$dq^*/dt = -(\partial G/\partial t)/(\partial G/\partial q) = -1 / (-2w^{2/3}r^{1/3}) = -1 / [2w^{2/3}r^{1/3}] < 0$

A per-unit tax unambiguously reduces output. The magnitude $1/[2w^{2/3}r^{1/3}]$ is larger when factor costs are low (flatter MC curve) and smaller when factor costs are high (steeper MC curve).

Master Reference Table

Property	Cobb-Douglas	Leontief	Linear	CES
MRTS	$(\beta/\alpha)(K/L)$	0 or ∞ (kink)	b/a (const)	$(\beta/\alpha)(K/L)^{1-\rho}$
Elasticity σ	1	0	∞	$1/(1-\rho)$
Cost function	$B \cdot w^{\beta/\Omega}r^{\alpha/\Omega}q^{1/\Omega}$	(wb+ra)·q	min(w/b,r/a)·q	$(\alpha^\sigma w^{1-\sigma}+...)^{1/(1-\sigma)}q$
RTS control	α+β vs 1	Always CRS	Always CRS	Always CRS (std)

Substitution	Imperfect, ∂L*/∂r > 0	None: ∂L*/∂r = 0	Perfect, corner	σ-elastic
Solve method	Lagrangian FOC	Kink inspection	Corner rule	Lagrangian FOC

Table 6.1 — Unified comparison across all production function families.

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