Advanced Microeconomic Theory (Econ 101A)

The Exhaustive Compendium & Problem Set Bank

Module-by-Module Pedagogical Breakdown

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Spring 2026

Contents

1 Module 1: Consumer Space & Axioms of Choice

Before solving optimization problems, the student must master the topology of preference. An agent's preferences must be **Complete** (can rank any two bundles), **Transitive** (if $A \succ B$ and $B \succ C$, then $A \succ C$), and **Continuous**.

1.1 Pathological Utility Functions & Marshallian Demand

Students often memorize the Cobb-Douglas result. We must force them to optimize edge-cases.

Example 1.1: Quasilinear Preferences
Let $U(x, y) = v(x) + y$, specifically $U(x, y) = \ln(x) + y$. *Optimization:* The constraint is $p_x x + p_y y = m$. The Lagrangian is:

$$\mathcal{L} = \ln(x) + y + \lambda(m - p_x x - p_y y)$$

FOC for x: $1/x - \lambda p_x = 0 \implies \lambda = 1/(xp_x)$.
FOC for y: $1 - \lambda p_y = 0 \implies \lambda = 1/p_y$.
Equating λ: $1/(xp_x) = 1/p_y \implies x^*(p_x, p_y, m) = p_y/p_x$.
Pedagogical Insight: Notice that Marshallian demand for x is completely independent of income m. All additional wealth is funneled purely into good y. This models "needs" vs "luxuries."

Example 1.2: Perfect Complements (Leontief)
Let $U(x, y) = \min(2x, y)$. *Optimization:* Calculus fails here because the indifference curves are L-shaped and non-differentiable at the kink. The consumer optimally consumes precisely at the kink: $2x = y$. Substitute into the budget constraint: $p_x x + p_y(2x) = m$.

$$x^*(p_x, p_y, m) = \frac{m}{p_x + 2p_y}, \quad y^*(p_x, p_y, m) = \frac{2m}{p_x + 2p_y}$$

Example 1.3: Perfect Substitutes
Let $U(x, y) = ax + by$. *Optimization:* This results in a corner solution. The consumer compares the "bang for the buck" for each good: $MU_x/p_x = a/p_x$ vs $MU_y/p_y = b/p_y$. If $a/p_x > b/p_y$, they spend identically everything on x: $x^* = m/p_x, y^* = 0$.

2 Utility Optimization Problem Bank (Berkeley Tier)

The following problem set acts as an escalation drill, forcing the student to abandon rote Lagrangian mechanicalism and utilize advanced economic logic (Kuhn-Tucker conditions, boundary constraints, and non-differentiability).

Tier 1: Easy (Standard Optimization)

The Setup: Given an asymmetric Cobb-Douglas utility function $U(x, y) = x^3 y$. The prices are $p_x = 2$, $p_y = 4$. The consumer has income $m = 120$. **Solution Engine:** $x^* = \frac{3m}{4p_x} = 45$ and $y^* = \frac{m}{4p_y} = 7.5$. $MRS = \frac{MU_x}{MU_y} = \frac{3y}{x} = 0.5$. The price ratio is $2/4 = 0.5$. The tangency condition flawlessly holds.

Tier 2: Medium (Quasilinear Corners & Wealth Gateways)

The Setup: A consumer has quasilinear utility $U(x, y) = 10\sqrt{x} + y$. Prices $p_x = 2, p_y = 5$. **Solution Engine:** FOC equations yield $MRS = \frac{5/\sqrt{x}}{1} = \frac{p_x}{p_y} \implies x^* = 156.25$. Notice x^* is a fixed constant, independent of wealth. Thus, $y^* = \frac{m-312.5}{5}$. If $m = 20$, the Lagrangian spits out a negative y, which is physically impossible. They hit a **corner solution**, spending 100% of wealth on x. Optimal bundle: $x = 10, y = 0$.

Tier 3: Advanced (The Bliss Point Constraint)

The Setup: Distance to a Bliss Point at $(10, 10)$: $U(x, y) = -(x - 10)^2 - (y - 10)^2$. Prices $p_x = 1, p_y = 1$. **Solution Engine:** If income is immensely high ($m = 1000$), the Marshallian demand is globally capped: $x^* = 10, y^* = 10$. The Lagrange multiplier λ (marginal utility of wealth) collapses to exactly 0. If given 15 units of x (food stamps) but $m = 4$, the consumer is pushed *past* satiation! To maximize utility, they use their \$4 cash entirely on y to get closer to $y = 10$. They consume $(15, 4)$.

3 Module 3: Duality & The Matrix of Revealed Preference

We now enter the Dual World (the Hal Varian approach).

3.1 The Exhaustive Duality Walkthrough (Cobb-Douglas)

Let $U(x, y) = x^{0.5}y^{0.5}$. Let's rigorously walk the bridge of Duality.

Step 1: The Indirect Utility Function $v(p, m)$
By substituting the Marshallian demands ($x^* = \frac{0.5m}{p_x}$) back into U:

$$v(p, m) = \frac{0.5m}{p_x^{0.5}p_y^{0.5}}$$

Step 2: The Expenditure Function $e(p, u)$
Because $v(p, m) = u$, we invert this identity to solve for m:

$$e(p, u) = 2up_x^{0.5}p_y^{0.5}$$

Step 3: Shephard's Lemma to Hicksian Demand
We extract compensated demand by differentiating $e(p, u)$ with respect to prices:

$$h_x(p, u) = \frac{\partial e}{\partial p_x} = u\left(\frac{p_y}{p_x}\right)^{0.5} \quad \text{and} \quad h_y(p, u) = u\left(\frac{p_x}{p_y}\right)^{0.5}$$

3.2 Revealed Preference (WARP & SARP)

Weak Axiom of Revealed Preference (WARP): If bundle A is purchased when B is affordable, $A \succ^D B$. If prices change, B can *never* be purchased if A is also affordable. Violating this means the consumer is mathematically irrational.

4 Module 4: Intertemporal Choice & Labor Supply

4.1 Two-Period Consumption Smoothing

An agent lives for $t = 0, 1$. $U(c_0, c_1) = \ln(c_0) + \beta \ln(c_1)$. Budget constraint: $c_0 + \frac{c_1}{1+R} = Y_0 + \frac{Y_1}{1+R} = W$. Equating the First Order Conditions yields the intertemporal Euler Equation:

$$\frac{c_1}{c_0} = \beta(1 + R)$$

4.2 Labor Supply: Income vs Substitution Events

Utility over consumption c and leisure l is $U(c, l) = c^\alpha l^{1-\alpha}$. Full Income Budget: $c + wl = wT + M$.

$$L^* = T - l^* = \alpha T - \frac{(1-\alpha)M}{w}$$

Because L^* has w in the denominator, $\partial L^*/\partial w > 0$. The substitution effect dynamically dominates the income effect.

5 Module 5: Producer Theory & Advanced Supply Mechanics

Let $y = L^{1/3}K^{1/3}$. **Step 1: Cost Minimization** Minimize $C = wL + rK$ subject to $L^{1/3}K^{1/3} = y$. The Expansion Path is $K = \frac{w}{r}L$. Substituting yields conditional factor demands:

$$L^*(w, r, y) = y^{1.5} \left(\frac{r}{w}\right)^{0.5}$$

Step 2: The Long-Run Cost Function

$$C(w, r, y) = wL^* + rK^* = 2y^{1.5}\sqrt{wr}$$

Step 3: Profit Maximization FOC ($p = MC$): $p = 3y^{0.5}\sqrt{wr} \implies y^*(p, w, r) = \frac{p^2}{9wr}$. This is the firm's uncompensated Supply Function.

6 Module 6: Extreme Examples & Game Theoretic Edge Cases

6.1 Asymmetric Information: The Akerlof Lemons Derivation

Quality $\theta \sim \text{Uniform}[0, 100]$. Sellers value their car at θ. Buyers value the cars at 1.5θ, but cannot observe θ. If the market price is P, the average quality *in the market* is $E[\theta|\theta \leq P] = \frac{P}{2}$. The maximum willing bid of a risk-neutral buyer is $1.5 \times \left(\frac{P}{2}\right) = 0.75P$. For a market to function, $Bid \geq P \implies 0.75P \geq P$. This inequality is only true when $P = 0$. The market completely unravels due to informational asymmetry.

6.2 Oligopoly: Stackelberg Leadership

Market Demand is $P = 120-(q_1+q_2)$. $c = 0$. Firm 2 (Follower) Best Response: $q_2^*(q_1) = 60-0.5q_1$. Firm 1 (Leader) maximizes: $\pi_1 = (120 - q_1 - (60 - 0.5q_1))q_1 \implies q_1^{Stack} = 60$. Firm 2 optimal output: $q_2 = 30$. Total Output: 90. The Leader secured an asymmetric profit (1800) compared to the follower (900).

7 Module 7: Advanced General Equilibrium (McKenzie)

Drawing upon the geometric methodologies of Lionel McKenzie's *Classical General Equilibrium Theory*.

7.1 The Edgeworth Box & The Contract Curve

Instead of partial equilibria, we assume a closed economy. Two agents (A, B), two goods (x, y). Endowments ω_A, ω_B. Pareto Efficiency requires the tangency of indifference curves. The **Contract Curve** is the locus of all allocations where $MRS_A = MRS_B$. For Cobb Douglas $U_A = x_A y_A$ and $U_B = x_B y_B$, with aggregate endowment $\bar{X}, \bar{Y}$:

$$\frac{y_A}{x_A} = \frac{y_B}{x_B} \implies \frac{y_A}{x_A} = \frac{\bar{Y} - y_A}{\bar{X} - x_A}$$

Solving this algebraically yields the curve spanning the endpoints of the Edgeworth Box.

7.2 The First & Second Fundamental Theorems of Welfare

First Theorem: If trade is voluntary and preferences are monotonic, any Walrasian competitive equilibrium is Pareto efficient. (It rests exactly on the Contract Curve). **Second Theorem (McKenzie's Separating Hyperplane):** If preferences and production sets are strictly convex, *any* Pareto efficient outcome on the Contract Curve can be achieved as a competitive equilibrium simply by redistributing the initial endowment ω and letting the market clear. Prices act as the separating hyperplane between the two convex upper-contour sets of the agents' utility functions.

8 Module 8: Externalities & Pigouvian Correction

Market failure structurally emerges when an agent's utility or production function contains arguments controlled by external entities.

8.1 The Mathematics of Missing Markets

Firm 1 produces steel s and pollution x. $C_1(s,x) = s^2 + (x-2)^2$. Firm 2 produces fish f but is damaged by x. $C_2(f,x) = f^2 + 2x$. Prices are p_s, p_f. **Private Equilibrium:** Firm 1 minimizes $(x-2)^2 \implies x = 2$ (maximizes pollution to save costs). **Social Optimum:** A central planner minimizes joint costs $C_{Total} = s^2 + (x-2)^2 + f^2 + 2x$. FOC with respect to x: $2(x-2) + 2 = 0 \implies x = 1$. The competitive market over-pollutes. **Pigouvian Tax:** The government applies a tax t on x. Firm 1's cost becomes $C_1 = s^2 + (x-2)^2 + tx$. FOC: $2(x-2)+t = 0$. To force $x = 1$, we set the tax exactly equal to the marginal external damage: $t = 2$.

9 Module 9: Public Goods & Free Riding

A pure public good G is non-rival and non-excludable. Let private good be x. $U_i(x_i, G) = \ln(x_i) + \ln(G)$. Endowment is w_i. Prices are $p_x = 1, p_G = 1$. $G = g_A + g_B$ (the sum of private contributions).

9.1 Nash Equilibrium (Private Provision)

Agent A maximizes $\ln(w_A - g_A) + \ln(g_A + g_B)$. FOC yields the Best Response function: $g_A^*(g_B) = \frac{w_A - g_B}{2}$. If the agents are identical ($w_A = w_B = w$), $g_A = g_B = g$. $g = \frac{w-g}{2} \implies g = w/3$. Total $G = 2w/3$.

9.2 The Samuelson Condition (Social Optimum)

A planner maximizes $\sum U$ subject to $\sum x_i + G = \sum w_i$. The Samuelson condition requires the sum of Marginal Rates of Substitution to equal the Marginal Rate of Transformation (MRT = 1).

$$MRS_A + MRS_B = MRT \implies \frac{x_A}{G} + \frac{x_B}{G} = 1 \implies \frac{w_A - g_A + w_B - g_B}{G} = 1$$

This yields $G_{optimal} = w$. The private market severely under-provides ($2w/3 < w$) because both agents attempt to free-ride on the other's contribution.

10 Module 10: Expected Utility & Risk Aversion

Drawing exactly on Hal Varian's state-preference paradigm.

10.1 The Arrow-Pratt Measures of Risk

If Utility of wealth is $u(w)$ where $u' > 0$ and $u'' < 0$, the agent is risk-averse. The coefficient of Absolute Risk Aversion is $ARA(w) = -\frac{u''(w)}{u'(w)}$. For Constant Absolute Risk Aversion (CARA) $u(w) = -e^{-\gamma w}$, $ARA = \gamma$.

10.2 Expected Utility and The Certainty Equivalent

A gamble pays \$100 with $p = 0.5$ and \$0 with $p = 0.5$. The Expected Value is \$50. Let $u(w) = \sqrt{w}$. The Expected Utility $E[U] = 0.5\sqrt{100} + 0.5\sqrt{0} = 5$. The **Certainty Equivalent** (CE) is the guaranteed cash that yields exact utility $E[U]$:

$$\sqrt{CE} = 5 \implies CE = \$25$$

The **Risk Premium** is the difference between the Expected Value and the Certainty Equivalent: $\$50 - \$25 = \$25$. A risk-averse consumer would pay up to \$25 to an insurance company to avoid taking this gamble.