Econ 101A

Final Exam Solutions Manual

10 Exam-Caliber Problems · Complete Step-by-Step Solutions

Dr. Ian Helfrich Helfrich Engine Tutoring

UC Berkeley Econ 101A · Spring 2026 · Prof. Cécile Gaubert

Problem Index

1 General Equilibrium (GE)
2 Cournot-Nash Equilibrium
3 Stackelberg Leadership
4 Negative Externality & Pigouvian Tax
5 Adverse Selection (Lemons)
6 Risk & CARA-Normal
7 Third-Degree Price Discrimination
8 IFT Comparative Statics
9 Slutsky Decomposition
10 Robinson Crusoe Synthesis

Contents

1 General Equilibrium in a Pure Exchange Economy 2
2 Cournot-Nash Equilibrium 5
3 Stackelberg Leadership (Sequential Quantity Competition) 7
4 Negative Externality & Pigouvian Tax 9
5 Adverse Selection (Akerlof Lemons Model) 11
6 Risk, Expected Utility, and CARA-Normal 13
7 Third-Degree Price Discrimination 15
8 IFT Comparative Statics 17
9 Slutsky Decomposition 19
10 Robinson Crusoe: The Synthesis Problem 21

1 General Equilibrium in a Pure Exchange Economy

Problem Statement

Two consumers (A, B) trade two goods (x, y) in a pure exchange economy.

$$U_A(x_A, y_A) = x_A^{1/2} y_A^{1/2}, \quad \omega_A = (4, 0); \qquad U_B(x_B, y_B) = x_B^{1/3} y_B^{2/3}, \quad \omega_B = (0, 3).$$

Normalize $p_y = 1$; let $p \equiv p_x$.

(A) Derive each consumer's Marshallian demands as functions of p.
(B) Find the Walrasian equilibrium price p^* and allocation.
(C) Verify Walras' Law: market y clears automatically.
(D) Confirm Pareto efficiency: $MRS_A = MRS_B = p^*$.
(E) **Edge case.** Generalize A's utility to $U_A = x_A^{\alpha} y_A^{1-\alpha}$. Find $p^*(\alpha)$ and show what happens as $\alpha \to 0$.

Part (A): Marshallian Demands

Consumer A — Cobb-Douglas, $\alpha = 1/2$

Wealth: $m_A = p \cdot \omega_x^A + 1 \cdot \omega_y^A = 4p$.
Income-share rule for $U_A = x^{1/2} y^{1/2}$ (equal exponents $\Rightarrow$ equal shares):

$$x_A^*(p) = \frac{\frac{1}{2} \cdot m_A}{p} = \frac{\frac{1}{2}(4p)}{p} = \mathbf{2}, \qquad y_A^*(p) = \frac{\frac{1}{2} \cdot m_A}{1} = \frac{\frac{1}{2}(4p)}{1} = \mathbf{2p}.$$

Note: $x_A^* = 2$ is constant in p. Why? A's endowment is entirely in x; as p rises, A's wealth rises proportionally, exactly offsetting the price increase.

Consumer B — Cobb-Douglas, $\alpha = 1/3$

Wealth: $m_B = p \cdot 0 + 1 \cdot 3 = 3$.
Income-share rule for $U_B = x^{1/3} y^{2/3}$:

$$x_B^*(p) = \frac{\frac{1}{3} \cdot 3}{p} = \frac{1}{p}, \qquad y_B^*(p) = \frac{\frac{2}{3} \cdot 3}{1} = \mathbf{2}.$$

Note: $y_B^* = 2$ is constant. B holds only y endowment; the same logic applies.

Part (B): Walrasian Equilibrium

Market Clearing for Good x

$$x_A^*(p^*) + x_B^*(p^*) = \omega_x^A + \omega_x^B \implies 2 + \frac{1}{p^*} = 4.$$

$$\frac{1}{p^*} = 2 \implies \boxed{p^* = \tfrac{1}{2}.}$$

Equilibrium allocation:

$$x_A^* = 2, \quad y_A^* = 2p^* = 1; \qquad x_B^* = 1/p^* = 2, \quad y_B^* = 2.$$

Total x: $2 + 2 = 4 = \omega_x$. Total y: $1 + 2 = 3 = \omega_y$. ✓

Part (C): Walras' Law

Market y Clears Automatically

$y_A^* + y_B^* = 1 + 2 = 3 = \omega_y^A + \omega_y^B = 0 + 3$. ✓
Why this must hold. Walras' Law: $\sum_i (p \cdot x^{i*} - p \cdot \omega^i) = 0$ by budget balance. If market x clears, market y must also clear. We checked it numerically as a sanity test.

Part (D): Pareto Efficiency Check

$MRS_A = MRS_B = p^*$

For $U_A = x^{1/2} y^{1/2}$:

$$MRS_A = \frac{MU_x^A}{MU_y^A} = \frac{\frac{1}{2} x_A^{-1/2} y_A^{1/2}}{\frac{1}{2} x_A^{1/2} y_A^{-1/2}} = \frac{y_A}{x_A} = \frac{1}{2} = p^*. \quad ✓$$

For $U_B = x^{1/3} y^{2/3}$:

$$MRS_B = \frac{MU_x^B}{MU_y^B} = \frac{\frac{1}{3} x_B^{-2/3} y_B^{2/3}}{\frac{2}{3} x_B^{1/3} y_B^{-1/3}} = \frac{y_B}{2x_B} = \frac{2}{2 \cdot 2} = \frac{1}{2} = p^*. \quad ✓$$

Both MRSs equal the equilibrium price ratio. The allocation is on the contract curve: no further trade is mutually beneficial. This is the **First Welfare Theorem** in action.

Part (E): Edge Case — $\alpha \to 0$

General $U_A = x_A^{\alpha} y_A^{1-\alpha}$

$m_A = 4p$ as before.

$$x_A^*(p) = \frac{\alpha \cdot 4p}{p} = 4\alpha, \qquad y_A^*(p) = (1-\alpha)\cdot 4p.$$

Market clearing:

$$4\alpha + \frac{1}{p^*} = 4 \implies \frac{1}{p^*} = 4(1-\alpha) \implies \boxed{p^*(\alpha) = \frac{1}{4(1-\alpha)} = \frac{\alpha}{1-\alpha}\cdot\frac{1}{4\alpha}.}$$

Wait — cleaner: $p^* = \dfrac{1}{4(1-\alpha)}$.

As $\alpha \to 0$: $p^* \to 1/4$. Consumer A barely wants x and holds all of it, so A dumps x on the market aggressively. x is cheap. $x_A^* = 4\alpha \to 0$ (A keeps almost no x), $x_B^* = 4(1-\alpha) \to 4$ (B gets nearly all of x).

As $\alpha \to 1$: $p^* \to \infty$. A hoards x; it becomes infinitely scarce from B's perspective.

2 Cournot-Nash Equilibrium

Problem Statement

Two firms produce a homogeneous good. Inverse demand: $P = 90 - Q$, $Q = q_1 + q_2$. Each firm has constant marginal cost $c = 15$.

(A) Derive each firm's best-response function.
(B) Solve for the symmetric Cournot-Nash equilibrium $(q_1^*, q_2^*, P^*, \pi_i^*)$.
(C) Verify the SOC.
(D) Compare to monopoly and perfect competition in a single table.
(E) With n symmetric firms, show that $P^* \to c$ as $n \to \infty$.

Part (A): Best-Response Functions

Firm 1's Problem (hold q_2 fixed)

$$\max_{q_1} \pi_1 = (P - c)q_1 = [90 - q_1 - q_2 - 15]q_1 = (75 - q_1 - q_2)q_1.$$

FOC (w.r.t. q_1, treating q_2 as a constant):

$$75 - 2q_1 - q_2 = 0 \implies q_1^{\text{BR}}(q_2) = \frac{75 - q_2}{2}.$$

Critical rule: differentiate *first*; impose symmetry $q_1 = q_2$ only after. By symmetry: $q_2^{\text{BR}}(q_1) = \frac{75 - q_1}{2}$.

Part (B): Nash Equilibrium

Solving the Symmetric System

Impose $q_1^* = q_2^* \equiv q^*$:

$$q^* = \frac{75 - q^*}{2} \implies 2q^* = 75 - q^* \implies 3q^* = 75 \implies \boxed{q^* = 25.}$$

$$Q^* = 50, \quad P^* = 90 - 50 = 40, \quad \pi_i^* = (40 - 15) \cdot 25 = 625.$$

Part (C): Second-Order Condition

SOC Verification

$$\frac{\partial^2 \pi_1}{\partial q_1^2} = -2 < 0. \quad \checkmark$$

The profit function is strictly concave in q_1: the FOC identifies a global maximum.

Part (D): Market Structure Comparison

Monopoly, Cournot, Perfect Competition

Monopoly: $MR = 90 - 2Q = 15 \implies Q^M = 37.5$, $P^M = 52.5$, $\pi^M = 1406.25$.
Perfect comp.: $P = c = 15$, $Q^{PC} = 75$, $\pi = 0$.

Structure	Q	P	π (industry)
Perfect competition	75	15	0
Cournot duopoly	50	40	1,250
Monopoly	37.5	52.5	1,406.25

Output: PC > Cournot > Monopoly. Price: PC < Cournot < Monopoly. ✓

Part (E): n-Firm Convergence

n Symmetric Firms

Each firm i: $\pi_i = (75 - q_i - Q_{-i})q_i$ where $Q_{-i} = (n-1)q^*$ by symmetry.
FOC: $75 - 2q_i^* - (n-1)q^* = 0$. With $q_i^* = q^*$: $75 - 2q^* - (n-1)q^* = 0 \implies q^*(n+1) = 75$.

$$q_i^* = \frac{75}{n+1}, \quad Q^* = \frac{75n}{n+1}, \quad P^* = 90 - \frac{75n}{n+1} = 15 + \frac{75}{n+1}.$$

As $n \to \infty$: $P^* \to 15 = c$ and $\pi_i^* \to 0$. *Cournot converges to perfect competition as the number of firms grows without bound.*

3 Stackelberg Leadership (Sequential Quantity Competition)

Problem Statement

Same market: $P = 90 - Q$, $c = 15$. Now Firm 1 **moves first** (leader); Firm 2 observes q_1 and then chooses q_2 (follower). Use backward induction.

(A) Solve the follower's best response $q_2^*(q_1)$.
(B) Firm 1 internalizes the follower's response. Derive and solve Firm 1's problem.
(C) Compute $(q_1^S, q_2^S, P^S, \pi_1^S, \pi_2^S)$.
(D) Compare to Cournot. Who gains and who loses from sequential play?
(E) **Key intuition**: why does the leader produce *more* than in Cournot?

Parts (A)–(C): Backward Induction Solution

Step 1 — Follower's Best Response (same as Cournot BR)

$q_2^*(q_1) = \dfrac{75 - q_1}{2}$.

Step 2 — Leader Substitutes the Follower's Response

Firm 1 treats q_2 not as a constant but as a *function* of its own choice:

$$\pi_1 = \left[90 - q_1 - \frac{75 - q_1}{2} - 15\right] q_1 = \left[75 - q_1 - \frac{75 - q_1}{2}\right] q_1 = \left[\frac{75 - q_1}{2}\right] q_1 = \frac{75q_1 - q_1^2}{2}.$$

FOC:

$$\frac{d\pi_1}{dq_1} = \frac{75 - 2q_1}{2} = 0 \implies \boxed{q_1^S = 37.5.}$$

SOC: $d^2\pi_1/dq_1^2 = -1 < 0$. ✓

Step 3 — Complete the Equilibrium

$$q_2^S = \frac{75 - 37.5}{2} = 18.75, \quad Q^S = 56.25, \quad P^S = 90 - 56.25 = 33.75.$$

$$\pi_1^S = (33.75 - 15)(37.5) = 18.75 \times 37.5 = 703.125.$$

$$\pi_2^S = (33.75 - 15)(18.75) = 18.75 \times 18.75 = 351.5625.$$

Part (D): Comparison to Cournot

Who Gains, Who Loses?

	q_1	q_2	π_1	π_2
Cournot (simultaneous)	25	25	625	625
Stackelberg (Firm 1 leads)	37.5	18.75	703.125	351.5625
Change	+12.5	−6.25	+78.125	−273.4375

Leader gains ($703 > 625$). **Follower loses** ($352 < 625$). Total industry profit $= 1{,}054.7 < 1{,}250$ (Cournot): sequential play expands total output, pushing price down and reducing joint profits.

Part (E): The Commitment Intuition

Why the Leader Produces More

In Cournot, if Firm 1 expands q_1 by 1 unit, total output rises by 1 full unit (Firm 2 doesn't react). In Stackelberg, if Firm 1 expands q_1 by 1 unit, Firm 2 *contracts* q_2 by 1/2 unit (its BR slope is $-1/2$). Total output rises by only 1/2 unit. The price impact is *smaller*. So the marginal cost of expanding output is lower for the Stackelberg leader — it's profitable to produce more. Formally: Firm 1's effective MR in Stackelberg accounts for $\partial q_2^*/\partial q_1 = -1/2$, making it less steep. Higher quantity is optimal.

4 Negative Externality & Pigouvian Tax

Problem Statement

A factory produces steel s at private cost $C_S(s) = s^2/2$, sells at price $p = 10$. Each unit of steel generates external damage $D(s) = s^2/4$ on a downstream fishery.

(A) Find the private (laissez-faire) optimum s^P.
(B) Find the social planner's optimum s^S.
(C) Find the Pigouvian per-unit tax t^* that implements s^S.
(D) Verify: under the tax, the firm privately chooses s^S.
(E) Calculate the deadweight loss of laissez-faire via integration.

Part (A): Private Optimum

Firm Ignores External Damage

$$\max_s \; ps - C_S(s) = 10s - \frac{s^2}{2}.$$

FOC: $10 - s = 0 \implies \boxed{s^P = 10.}$ SOC: $-1 < 0.$✓

Part (B): Social Optimum

Planner Maximizes Total Surplus

Social surplus $= \pi_S - D(s) = 10s - \frac{s^2}{2} - \frac{s^2}{4} = 10s - \frac{3s^2}{4}$.

$$\text{FOC:} \quad 10 - \frac{3s}{2} = 0 \implies \boxed{s^S = \frac{20}{3} \approx 6.67.}$$

The social optimum requires *less* production ($s^S < s^P$) because the planner internalizes the fishery's damage.

Interpretation of the social FOC. The full marginal social cost is:

$$MSC(s) = C_S'(s) + D'(s) = s + \frac{s}{2} = \frac{3s}{2}.$$

Set $MSC = p$: $\frac{3s}{2} = 10 \implies s^S = \frac{20}{3}$.

Part (C): Pigouvian Tax

$t^* =$ Marginal Damage at s^S

The Pigouvian tax equals the *marginal external damage evaluated at the social optimum*:

$$t^* = D'(s^S) = \frac{s^S}{2} = \frac{1}{2} \cdot \frac{20}{3} = \boxed{\frac{10}{3} \approx 3.33.}$$

Part (D): Verification Under the Tax

Tax-Augmented Private FOC

Firm faces effective net price $(p - t^*)$ per unit:

$$\max_s \, (p - t^*)s - \frac{s^2}{2}.$$

FOC: $p - t^* - s = 0 \implies s = 10 - \frac{10}{3} = \frac{20}{3} = s^S$. ✓
The tax exactly offsets the externality: the firm's private incentive now aligns with the social planner's.

Part (E): Deadweight Loss

DWL = Integral of Excess Social Cost

DWL is the welfare loss from producing too much ($s^P - s^S$ excess units), where each unit's social cost exceeds its benefit:

$$\text{DWL} = \int_{s^S}^{s^P} \big[MSC(s) - p\big]\, ds = \int_{20/3}^{10} \left(\frac{3s}{2} - 10\right) ds.$$

$$= \left[\frac{3s^2}{4} - 10s\right]_{20/3}^{10} = (75 - 100) - \left(\frac{3 \cdot (20/3)^2}{4} - \frac{200}{3}\right).$$

$$= (-25) - \left(\frac{3 \cdot 400/9}{4} - \frac{200}{3}\right) = (-25) - \left(\frac{100}{3} - \frac{200}{3}\right) = (-25) - \left(-\frac{100}{3}\right) = -25 + \frac{100}{3} = \frac{-75 + 100}{3} = \boxed{\frac{25}{3} \approx 8.3}$$

5 Adverse Selection (Akerlof Lemons Model)

Problem Statement

A used-car market has two quality types. Fraction q of cars are good; fraction $1 - q$ are lemons.

Type	Seller's value	Buyer's value
Lemon	\$4,000	\$6,000
Good	\$8,000	\$12,000

Sellers know their car's quality; buyers cannot observe it. The seller has all bargaining power ($P^* =$ buyer's WTP).

(A) Write buyer's WTP as a function of belief $\mu = \Pr(\text{good} \mid P)$.
(B) Derive each seller type's participation condition.
(C) Find the pooling equilibrium. What constraint on q does it require?
(D) Show that if $q < 1/3$, only the lemons equilibrium exists. Describe it.
(E) What is the social cost of the lemons equilibrium? Is this outcome Pareto efficient?

Part (A): Buyer's Willingness to Pay

Expected Car Value for the Buyer

Buyer values a car at \$12,000 if good (prob μ) and \$6,000 if lemon (prob $1 - \mu$):

$$P^*(\mu) = \mu \cdot 12{,}000 + (1 - \mu) \cdot 6{,}000 = 6{,}000 + 6{,}000\mu.$$

P ranges from \$6,000 (all lemons, $\mu = 0$) to \$12,000 (all good, $\mu = 1$).

Part (B): Seller Participation Conditions

Who Sells at Price $P^*(\mu)$?

Lemon seller participates if $P^* \geq 4{,}000$: $6{,}000 + 6{,}000\mu \geq 4{,}000$ — always true for any $\mu \geq 0$.
Lemons always sell.
Good-car seller participates if $P^* \geq 8{,}000$:

$$6{,}000 + 6{,}000\mu \geq 8{,}000 \implies \mu \geq \frac{2{,}000}{6{,}000} = \frac{1}{3}.$$

Good cars are only offered when buyers believe at least 1/3 of offered cars are good.

Part (C): Pooling Equilibrium

Both Types Sell; Belief Must Be Self-Consistent

In a pooling equilibrium, all cars are offered, so the fraction of good cars among those for sale equals the true fraction: $\mu = q$.
Condition for pooling: good-car sellers participate, i.e., $q \geq 1/3$.
Pooling equilibrium price: $P^* = 6{,}000 + 6{,}000q$.

Example: $q = 1/2 \implies P^* = \$9{,}000$. Both types sell. All gains from trade realized.

Part (D): Lemons Equilibrium when $q < 1/3$

Market Unraveling

If $q < 1/3$: at the pooling price $P^* = 6{,}000 + 6{,}000q < 8{,}000$, good-car owners refuse to sell.
Buyers anticipate this: if good cars never appear, $\mu = 0$, so $P^* = \$6{,}000$.
At $P^* = 6{,}000$, good-car owners confirm they won't sell ($6{,}000 < 8{,}000$). *Beliefs are self-consistent.*
Lemons equilibrium: $\mu = 0$, $P^* = \$6{,}000$. *Only lemons trade.*

The mechanism: a low price → good sellers exit → average quality falls → buyers lower WTP → price stays low. Self-reinforcing spiral.

Part (E): Social Cost

Welfare Loss from Adverse Selection

In the lemons equilibrium, no good cars are traded.
Each good car generates gains from trade of $\$12{,}000 - \$8{,}000 = \$4{,}000$ (buyer's value minus seller's reservation). These gains are destroyed.
The outcome is not Pareto efficient: good-car owners and buyers could both be made better off if good cars traded, but the information structure prevents it. No fraud is occurring — the inefficiency arises purely from the information asymmetry.
This is why the First Welfare Theorem fails here: the market for good cars is missing.

6 Risk, Expected Utility, and CARA-Normal

Problem Statement

An investor has CARA utility $U(w) = -e^{-\gamma w}$ with $\gamma = 0.02$. Initial wealth $W_0 = 100$. An investment costs $I = 10$ upfront and yields a stochastic return $\tilde{X} \sim N(25, 400)$ (mean $\mu_X = 25$, variance $\sigma_X^2 = 400$).

(A) Prove strict risk aversion and show ARA $= \gamma$.

(B) If the investor takes the investment, what is the distribution of final wealth $\tilde{W}$?

(C) Compute the Certainty Equivalent (CE) using the CARA-Normal formula.

(D) Under what condition does the investor invest? Apply it here.

(E) As $\gamma \to 0$ (risk-neutral limit) and $\gamma \to \infty$ (infinitely risk averse), describe the investment decision.

Part (A): Risk Aversion and ARA

Concavity and Arrow-Pratt Measure

$$U'(w) = \gamma e^{-\gamma w} > 0, \qquad U''(w) = -\gamma^2 e^{-\gamma w} < 0.$$

$U'' < 0$ confirms **strict concavity** $\Rightarrow$ strict risk aversion. By Jensen's Inequality: $E[U(\tilde{w})] < U(E[\tilde{w}])$.

Arrow-Pratt absolute risk aversion:

$$ARA(w) = -\frac{U''(w)}{U'(w)} = \frac{\gamma^2 e^{-\gamma w}}{\gamma e^{-\gamma w}} = \gamma.$$

Constant in w: this is the defining property of CARA (Constant Absolute Risk Aversion).

Part (B): Distribution of Final Wealth

$\tilde{W} = W_0 - I + \tilde{X}$

$$\tilde{W} = 100 - 10 + \tilde{X} = 90 + \tilde{X}.$$

Since $\tilde{X} \sim N(25, 400)$ and the shift is by a constant:

$$\tilde{W} \sim N(90 + 25,\ 400) = N(115, 400).$$

$E[\tilde{W}] = 115$, $\text{Var}(\tilde{W}) = 400$, $\text{SD}(\tilde{W}) = 20$.

Part (C): Certainty Equivalent via CARA-Normal Formula

Derivation from the Moment-Generating Function

For $\tilde{W} \sim N(\mu_W, \sigma_W^2)$:

$$E[U(\tilde{W})] = E[-e^{-\gamma \tilde{W}}] = -e^{-\gamma\mu_W + \frac{1}{2}\gamma^2\sigma_W^2} \quad \text{(MGF of Normal with } t = -\gamma\text{)}.$$

Set $U(CE) = E[U(\tilde{W})]$:

$$-e^{-\gamma\, CE} = -e^{-\gamma\mu_W + \frac{1}{2}\gamma^2\sigma_W^2} \implies -\gamma\, CE = -\gamma\mu_W + \tfrac{1}{2}\gamma^2\sigma_W^2.$$

$$\boxed{CE = \mu_W - \frac{\gamma}{2}\sigma_W^2.}$$

Numerically: $CE = 115 - \dfrac{0.02}{2} \cdot 400 = 115 - 4 = \mathbf{111}$.

Risk premium $= E[\tilde{W}] - CE = 115 - 111 = 4$. The investor would pay up to \$4 to eliminate the risk entirely.

Part (D): Investment Condition

Invest iff $CE > W_0$

The investor takes the investment if and only if $CE > W_0$ (the risky option is preferred to the status quo):

$$CE = 111 > 100 = W_0. \quad \checkmark \quad \text{Invest.}$$

General condition (substitute CE formula):

$$\mu_X - I - \frac{\gamma}{2}\sigma_X^2 > 0 \implies 25 - 10 - \frac{0.02}{2}(400) = 25 - 10 - 4 = 11 > 0. \quad \checkmark$$

The "adjusted net return" $\mu_X - I$ must exceed the risk-loading $\frac{\gamma}{2}\sigma_X^2$.

Part (E): Limiting Cases

Risk-Neutral and Infinitely Risk-Averse Limits

As $\gamma \to 0$: $CE \to \mu_W = 115 > 100$. The investor decides purely on expected value. Invest whenever $E[\tilde{X}] > I$ (here: $25 > 10$). ✓

As $\gamma \to \infty$: $CE \to -\infty$. The risk penalty $\frac{\gamma}{2}\sigma^2$ grows without bound. No investment is ever worth it. The infinitely risk-averse investor holds only the safe asset.

7 Third-Degree Price Discrimination

Problem Statement

A monopolist sells in two segmented markets with $MC = 20$ (constant).

$$\text{Market 1 (students): } P_1 = 100 - 2Q_1$$
$$\text{Market 2 (professionals): } P_2 = 160 - Q_2$$

(A) Find the profit-maximizing quantities and prices in each market.
(B) Verify using the inverse-elasticity (Lerner) rule.
(C) Compute discriminating profit π_D.
(D) Find the optimal uniform price P^U and profit π_U. Quantify the gain from discrimination.

Part (A): Optimal Quantities and Prices

Set $MR_i = MC$ in Each Market Independently

Market 1: $MR_1 = 100 - 4Q_1$. Set $= 20$:

$$100 - 4Q_1 = 20 \implies Q_1^* = 20, \quad P_1^* = 100 - 2(20) = \mathbf{60}.$$

Market 2: $MR_2 = 160 - 2Q_2$. Set $= 20$:

$$160 - 2Q_2 = 20 \implies Q_2^* = 70, \quad P_2^* = 160 - 70 = \mathbf{90}.$$

Market 2 (professionals) pays a higher price ($90 vs. $60).

Part (B): Inverse-Elasticity Verification

Lerner Index: $(P - MC)/P = 1/|\varepsilon|$

Market 1: $dQ_1/dP_1 = -1/2$.

$$\varepsilon_1 = \frac{dQ_1}{dP_1} \cdot \frac{P_1}{Q_1} = -\frac{1}{2} \cdot \frac{60}{20} = -\frac{3}{2} \implies |\varepsilon_1| = \frac{3}{2}.$$

Lerner check: $\dfrac{P_1^* - MC}{P_1^*} = \dfrac{40}{60} = \dfrac{2}{3} = \dfrac{1}{3/2}$. ✓

Market 2: $dQ_2/dP_2 = -1$.

$$\varepsilon_2 = -1 \cdot \frac{90}{70} = -\frac{9}{7} \implies |\varepsilon_2| = \frac{9}{7}.$$

Lerner check: $\dfrac{90 - 20}{90} = \dfrac{7}{9} = \dfrac{1}{9/7}$. ✓

Key result: $|\varepsilon_2| = 9/7 < 3/2 = |\varepsilon_1|$. Market 2 is *less elastic* ⇒ higher price. The firm extracts more surplus where demand is less responsive.

Part (C): Discriminating Profit

$\pi_D = \sum_i (P_i^* - MC)\, Q_i^*$

$$\pi_D = (60 - 20)(20) + (90 - 20)(70) = 800 + 4{,}900 = \$5{,}700.$$

Part (D): Uniform Price Benchmark

Find Uniform P^U That Maximizes $\pi = (P - 20)Q$

For $P \leq 100$ both markets are served:

$$Q_1(P) = \frac{100 - P}{2}, \qquad Q_2(P) = 160 - P.$$

$$Q(P) = \frac{100 - P}{2} + 160 - P = 210 - \frac{3P}{2}.$$

Inverse: $P = 140 - \frac{2Q}{3}$. MR$= 140 - \frac{4Q}{3}$. Set $= 20$:

$$140 - \tfrac{4Q}{3} = 20 \implies Q^* = 90, \quad P^U = 140 - 60 = 80.$$

Check: $P^U = 80 \leq 100$. Both markets active. $Q_1 = 10$, $Q_2 = 80$. ✓

$$\pi_U = (80 - 20)(90) = \$5{,}400.$$

Gain from price discrimination: $\pi_D - \pi_U = 5{,}700 - 5{,}400 = \$300$.

8 IFT Comparative Statics

Problem Statement

A firm has production function $f(L)$ with $f' > 0$, $f'' < 0$. It sells output at price p, pays wage w, and faces a non-linear tax: net revenue per unit of output is reduced by $\tau \cdot f(L)$, so the firm's profit is

$$\pi(L) = p\,f(L) - \tau\,[f(L)]^2 - w\,L.$$

(A) Derive the FOC that implicitly defines optimal labor $L^*(\tau)$.
(B) Verify the SOC.
(C) Use the **Implicit Function Theorem** to derive $\partial L^*/\partial \tau$.
(D) Sign the result definitively. What does the tax do to labor demand?
(E) Compute $\partial L^*/\partial p$ via IFT and interpret.

Part (A): First-Order Condition

FOC w.r.t. L

$$\frac{\partial \pi}{\partial L} = p\,f'(L) - 2\tau\,f(L)\,f'(L) - w = 0.$$

Factor out $f'(L)$:

$$f'(L)\big[p - 2\tau\,f(L)\big] - w = 0.$$

This is the FOC that *implicitly* defines L^*. We cannot solve for L^* in closed form without specifying f.

Part (B): Second-Order Condition

$\partial^2\pi/\partial L^2 < 0$ Required

$$\frac{\partial^2 \pi}{\partial L^2} = p\,f''(L) - 2\tau\big[(f'(L))^2 + f(L)f''(L)\big] = \big[p - 2\tau f(L)\big]f''(L) - 2\tau\,[f'(L)]^2.$$

At the optimum, $p - 2\tau f(L^*) > 0$ (from the FOC structure and $w > 0$, $f' > 0$). Both terms are negative: $[p - 2\tau f]f'' < 0$ since $f'' < 0$; and $-2\tau[f']^2 < 0$ since $\tau > 0$. Therefore $\partial^2\pi/\partial L^2 < 0$. ✓

Part (C): IFT Application

Define $F(L, \tau) = 0$ and Apply IFT

Let

$$F(L, \tau) \equiv f'(L)\big[p - 2\tau f(L)\big] - w = 0.$$

The **Implicit Function Theorem** gives:

$$\frac{\partial L^*}{\partial \tau} = -\frac{F_\tau}{F_L}.$$

Compute F_τ (differentiate F w.r.t. τ, holding L fixed):

$$F_\tau = f'(L) \cdot (-2f(L)) = -2f(L)f'(L).$$

Compute F_L (differentiate F w.r.t. L, holding τ fixed):

$$F_L = f''(L)\big[p - 2\tau f(L)\big] + f'(L) \cdot (-2\tau f'(L)) = \big[p - 2\tau f\big] f'' - 2\tau [f']^2 = \frac{\partial^2 \pi}{\partial L^2}.$$

Therefore:

$$\frac{\partial L^*}{\partial \tau} = -\frac{-2f(L^*)f'(L^*)}{F_L} = \frac{2f(L^*)f'(L^*)}{F_L}.$$

Part (D): Signing the Result

Direction of the Tax Effect

- Numerator: $2f(L^*)f'(L^*) > 0$ since $f > 0$ and $f' > 0$.
- Denominator: $F_L = \partial^2\pi/\partial L^2 < 0$ (SOC).

$$\frac{\partial L^*}{\partial \tau} = \frac{(+)}{(-)} < 0.$$

A higher tax reduces labor demand. Economic intuition: the quadratic tax penalizes output more as output grows, so the marginal return to hiring labor falls. The firm contracts.

Part (E): Effect of Output Price

$\partial L^*/\partial p$ via IFT

$F_p = f'(L) > 0$.

$$\frac{\partial L^*}{\partial p} = -\frac{F_p}{F_L} = -\frac{f'(L^*)}{(\text{negative})} = \frac{f'(L^*)}{|F_L|} > 0.$$

A higher output price increases labor demand. Higher p raises the marginal revenue product of labor, so the firm hires more.

9 Slutsky Decomposition

Problem Statement

Consumer has $U(x,y) = x^{1/2}y^{1/2}$, income $m = 20$. Initial prices: $p_x = 1$, $p_y = 1$. The price of x rises to $p'_x = 4$.

(A) Compute Marshallian demand before and after the price change.
(B) Derive the expenditure function $e(p_x, p_y, u)$ and Hicksian demand h_x.
(C) Decompose the total effect into substitution effect (SE) and income effect (IE).
(D) Verify the Slutsky equation holds numerically.
(E) Is x a normal good? Use your decomposition to confirm.

Part (A): Marshallian Demands

Before and After

For $U = x^{1/2}y^{1/2}$ (symmetric CD), spend half income on each good:

$$x^*(p_x, m) = \frac{m}{2p_x}, \qquad y^*(p_y, m) = \frac{m}{2p_y}.$$

Before ($p_x = 1$, $p_y = 1$, $m = 20$): $x_0^* = 10$, $y_0^* = 10$, $u_0 = \sqrt{10 \cdot 10} = 10$.
After ($p'_x = 4$, $m = 20$): $x_1^* = \dfrac{20}{2 \cdot 4} = 2.5$, $y_1^* = 10$ (unchanged).

Part (B): Expenditure Function and Hicksian Demand

Duality: Solve the Cost-Minimization Problem

Cost minimize: $\min p_x x + p_y y$ s.t. $x^{1/2}y^{1/2} = u$.

Tangency: $MRS = p_x/p_y \implies y/x = p_x/p_y \implies y = \frac{p_x}{p_y}x$. Sub into constraint: $x^{1/2}\left(\frac{p_x}{p_y}x\right)^{1/2} = u \implies x\left(\frac{p_x}{p_y}\right)^{1/2} = u$.

$$h_x(p_x, p_y, u) = u\sqrt{\frac{p_y}{p_x}}, \qquad h_y = u\sqrt{\frac{p_x}{p_y}}.$$

$$e(p_x, p_y, u) = p_x h_x + p_y h_y = u\sqrt{p_x p_y} + u\sqrt{p_x p_y} = 2u\sqrt{p_x p_y}.$$

Shephard's Lemma check: $\partial e/\partial p_x = 2u \cdot \frac{1}{2}\sqrt{p_y/p_x} = u\sqrt{p_y/p_x} = h_x$.✓

Part (C): Substitution Effect and Income Effect

Hicksian demand at new price, original utility

At $p'_x = 4$, $u_0 = 10$:

$$h_x(4, 1, 10) = 10\sqrt{\frac{1}{4}} = 10 \cdot \frac{1}{2} = 5.$$

$$\textbf{SE} = h_x(p'_x) - x^*_0 = 5 - 10 = \mathbf{-5}.$$

$$\textbf{IE} = x^*_1 - h_x(p'_x) = 2.5 - 5 = \mathbf{-2.5}.$$

$$\textbf{TE} = x^*_1 - x^*_0 = 2.5 - 10 = \mathbf{-7.5} = SE + IE = -5 + (-2.5). \quad \checkmark$$

Parts (D)–(E): Slutsky Equation Verification & Normality

Formula Verification

Slutsky equation: $\partial x^*/\partial p_x = \partial h_x/\partial p_x - x^* \cdot \partial x^*/\partial m$.

$$\frac{\partial h_x}{\partial p_x} = u \cdot \left(-\frac{1}{2}\right) p_x^{-3/2} p_y^{1/2} = -\frac{u}{2}\sqrt{\frac{p_y}{p_x^3}} = -\frac{10}{2}\sqrt{\frac{1}{4}} = -\frac{10}{4} = -2.5.$$

$$\frac{\partial x^*}{\partial m} = \frac{1}{2p_x} = \frac{1}{8}. \quad x^*_0 = 10 \text{ (at original prices? — use Hicksian point: } h_x = 5).$$

Actually using the standard form at the Hicksian point ($x = h_x = 5$, $p'_x = 4$): $\partial x^*/\partial p_x = -m/(2p_x^2) = -20/32 = -5/8$. Slutsky: $-2.5 - 5 \cdot (1/8) = -2.5 - 0.625 = -3.125$. (Rate per unit price change; multiply by $\Delta p_x = 3$ gives $-9.375\ldots$)

Simpler verification via finite differences:
$\Delta x_{\text{total}} = -7.5$, $\Delta p_x = 3$. SE$= -5$: a move along the indifference curve (utility held at $u_0 = 10$, price rises to 4). Pure substitution, no wealth change. IE$= -2.5$: the real-income loss from the price hike.
Normality: $\partial x^*/\partial m = 1/(2p_x) > 0$ for any $p_x > 0$. Good x is a **normal good**: IE and SE work in the *same direction* (both negative). Both effects reduce demand when price rises — law of demand holds with no ambiguity.

10 Robinson Crusoe: The Synthesis Problem

Problem Statement

Robinson Crusoe lives alone. He is simultaneously the only consumer and the only firm.

$$U(c,\ell) = \ln c + \ln \ell, \quad T = 24 \text{ (time endowment)}, \quad c = \sqrt{L} \text{ (production)}, \quad L = T - \ell.$$

(A) **Planner's problem.** Maximize U directly by choosing ℓ.
(B) **Firm's problem.** Let wage $= w$, coconut price $p_c = 1$. Find labor demand $L^d(w)$ and profit $\pi(w)$.
(C) **Consumer's problem.** Robinson receives wage income wT and profits $\pi(w)$. Find his optimal $(\ell^*(w), c^*(w))$.
(D) **Market clearing.** Set $L^d = T - \ell^*$ to find w^*.
(E) Verify the decentralized allocation = planner's allocation. State which welfare theorem this demonstrates.
(F) **Edge case.** If production were $c = L$ (linear, CRS), what is w^* and π^*?

Part (A): Planner's Problem

Substitute $c = \sqrt{T-\ell}$; Maximize Over ℓ

$$\max_{\ell} \ \ln\sqrt{T-\ell} + \ln \ell = \frac{1}{2}\ln(T-\ell) + \ln \ell.$$

FOC:

$$\frac{-1/2}{T-\ell^*} + \frac{1}{\ell^*} = 0 \implies \frac{1}{\ell^*} = \frac{1}{2(T-\ell^*)} \implies 2(T-\ell^*) = \ell^* \implies 2T = 3\ell^* \implies \ell^* = \frac{2T}{3} = 16.$$

$$L^* = T - \ell^* = \frac{T}{3} = 8, \qquad c^* = \sqrt{8} = 2\sqrt{2}.$$

SOC: $d^2U/d\ell^2 = -\frac{1}{4}(T-\ell)^{-2} - \ell^{-2} < 0.$ ✓

Part (B): Firm's Problem

Maximize $\pi = \sqrt{L} - wL$

$$\max_{L} \ \sqrt{L} - wL.$$

FOC: $\dfrac{1}{2\sqrt{L^d}} = w \implies L^d(w) = \dfrac{1}{4w^2}.$

SOC: $\partial^2\pi/\partial L^2 = -\frac{1}{4}L^{-3/2} < 0.$ ✓

Profit:

$$\pi(w) = \sqrt{L^d} - w \cdot L^d = \frac{1}{2w} - w \cdot \frac{1}{4w^2} = \frac{1}{2w} - \frac{1}{4w} = \frac{1}{4w}.$$

Part (C): Consumer's Problem

Income $= wT + \pi(w)$; Maximize $\ln c + \ln \ell$

$$m(w) = wT + \pi(w) = 24w + \frac{1}{4w}.$$

Budget constraint: $c + w\ell = m(w)$ (leisure costs w per unit of time).
Cobb-Douglas $\ln c + \ln \ell$ with equal weights $\Rightarrow$ spend half income on each:

$$c^*(w) = \frac{m(w)}{2} = 12w + \frac{1}{8w}, \qquad w\ell^*(w) = \frac{m(w)}{2} \implies \ell^*(w) = \frac{m(w)}{2w} = 12 + \frac{1}{8w^2}.$$

Part (D): Market Clearing — Find w^*

$L^d(w) = T - \ell^*(w)$

$$\frac{1}{4w^2} = 24 - \left(12 + \frac{1}{8w^2}\right) = 12 - \frac{1}{8w^2}.$$

$$\frac{1}{4w^2} + \frac{1}{8w^2} = 12 \implies \frac{2+1}{8w^2} = 12 \implies \frac{3}{8w^2} = 12 \implies w^{*2} = \frac{3}{96} = \frac{1}{32} \implies \boxed{w^* = \frac{1}{4\sqrt{2}}.}$$

Part (E): Verify Planner = Decentralized

First Welfare Theorem in Action

$$L^* = \frac{1}{4w^{*2}} = \frac{1}{4 \cdot (1/32)} = \frac{32}{4} = 8. \quad \checkmark \quad \text{(matches planner)}$$

$$\ell^* = T - L^* = 16. \quad \checkmark \qquad c^* = \sqrt{8} = 2\sqrt{2}. \quad \checkmark$$

The decentralized market, operating through the single price w^*, exactly replicates the social planner's allocation.

First Welfare Theorem: Every competitive equilibrium is Pareto efficient. Prices here are just *bookkeeping*: they split the single agent's brain into "firm" and "consumer," but both sides land at the same optimum. If they didn't, the agent would be disagreeing with itself — which is impossible.

Part (F): CRS Edge Case

$c = L$ (Linear Production)

Firm profit: $\pi = (1-w)L$. Three cases:

- $w < 1$: profit is $(1-w)L$ with $(1-w) > 0 \Rightarrow$ set $L \to \infty$. **No finite equilibrium.**
- $w > 1$: profit is negative for any $L > 0 \Rightarrow$ firm shuts down ($L = 0$). No production.
- $w = 1$: profit $= 0$ for any L. Firm is indifferent; supply is indeterminate.

Equilibrium requires $w^* = 1$, $\pi^* = 0$. This is the *zero-profit condition* that pins wages in any CRS industry: competition drives profits to zero and the wage equals the marginal product of labor ($= 1$ here).

Appendix — Solution Shortcuts at a Glance

The 10-Second Checklist for Each Problem Type

Problem Type	Checklist
GE	(1) Wealth $= p\omega_x + \omega_y$. (2) Demands. (3) $\sum x_i^* = \omega_x$, solve p^*. (4) Walras' Law to skip market 2. (5) $MRS_A = MRS_B = p^*$ for efficiency.
Cournot	(1) Write π_i. (2) FOC in q_i, hold q_{-i} fixed. (3) Impose symmetry *after*. (4) SOC $= -2b < 0$.✓
Stackelberg	(1) Solve follower BR. (2) Sub BR into leader's π_1. (3) FOC for leader. (4) Back out follower's q_2^*.
Externalities	(1) Private FOC: $p = MC_S$. (2) Social FOC: $p = MC_S + MD$. (3) Tax $= MD(q^S)$. (4) DWL $= \int_{q^S}^{q^P} (MSC - p)\,dq$.
Adverse Selection	(1) Buyer WTP(μ). (2) Good-seller participation $\Rightarrow$ threshold μ^*. (3) Pooling if $q \geq \mu^*$; lemons otherwise.
CARA Risk	(1) ARA $= -U''/U' = \gamma$. (2) Final wealth distribution. (3) CARA-Normal: $CE = \mu_W - \frac{\gamma}{2}\sigma_W^2$. (4) Invest iff $CE > W_0$.
Price Discrimination	(1) $MR_i = MC$ in each market. (2) Lerner: $(P_i - MC)/P_i = 1/\lvert\varepsilon_i\rvert$. (3) Less elastic $\Rightarrow$ higher price.
IFT Comp. Statics	(1) Write FOC $F(x, \alpha) = 0$. (2) $\partial x^*/\partial\alpha = -F_\alpha/F_x$. (3) Sign numerator and denominator separately.
Slutsky	(1) Compute Hicksian demand $h_x = u\sqrt{p_y/p_x}$. (2) SE$= h_x(p') - x_0^*$. (3) IE$= x_1^* - h_x(p')$. (4) Check SE+IE$=$ TE.
Robinson Crusoe	(1) Planner: sub $c = f(T - \ell)$, maximize U. (2) Firm: FOC $\Rightarrow L^d(w)$, profit $\pi(w)$. (3) Consumer: income $= wT + \pi$, demand. (4) Market clear $\Rightarrow w^*$. (5) Verify match.

Good luck, Vedant.

You have the tools. Now execute. Yuuuuuur.